On the Rate of Convergence to the Marchenko–Pastur Distribution

F. Götze
Faculty of Mathematics
University of Bielefeld
Germany

A. Tikhomirov
Department of Mathematics
Komi Research Center of Ural Branch of RAS,
Syktyvkar state University
Syktyvkar, Russia

Abstract

Let $\mathbf{X}=(X_{jk})$ denote $n\times p$ random matrix with entries X_{jk}, which are independent for $1\le j\le n, 1\le k\le p$. We consider the rate of convergence of empirical spectral distribution function of matrix $\mathbf{W}=\frac1p\mathbf{X}\mathbf{X}^*$ to the Marchenko–Pastur law. We assume that $\mathbf{E}X_{jk}=0$, $\mathbf{E}X_{jk}^2=1$ and that the distributions of the matrix elements X_{jk} have a uniformly sub exponential decay in the sense that there exists a constant $\varkappa>0$ such that for any $1\le j\le n$, $1\le k\le p$ and any $t\ge1$ we have

$$\Pr\{|X_{jk}|>t\}\le\varkappa^{-1}\exp\{-t^{\varkappa}\}.$$

By means of a recursion argument it is shown that the Kolmogorov distance between the empirical spectral distribution of the sample covariance matrix $\mathbf{W}$ and the Marchenko–Pastur distribution is of order $O(n^{-1}\log^b n)$ with some positive constant $b>0$.

1 Introduction

Consider a family of independent random variables $\mathbf{X}=\{X_{jk}\}$, $1\le j\le n, 1\le k\le p$, defined on some probability space $(\Omega,\mathfrak{M},\Pr)$. Let $\mathbf{X}=(X_{jk})$ be matrix of order $n\times p$ and let $\mathbf{W}=\frac1p\mathbf{X}\mathbf{X}^*$. Denote by $\{s_1^2,\dots,s_n^2\}$ the eigenvalues of the matrix $\mathbf{W}$ and introduce the associated spectral distribution function

$$\mathcal{F}_n(x)=\frac1n\,\mathrm{card}\,\{j\le n: s_j^2\le x\},\quad x\in\mathbb{R}.$$

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Averaging over the random values $X_{ij}(\omega)$, define the expected (non-random) empirical distribution functions

$$F_n(x) = \mathbf{E}\,\mathcal{F}_n(x).$$

We assume that $p = p(n)$ and $\lim_{n\to\infty} \frac{n}{p} = y \in (0, \infty)$. Without loss of generality we shall assume that $y \in (0, 1]$. Let $G_y(x)$ denote the Marchenko–Pastur distribution function with density $g_y(x) = G_y'(x) = \frac{1}{2yx\pi}\sqrt{(x-a)(b-x)}I_{[a,b]}(x)$, where $I_{[a,b]}(x)$ denotes the indicator–function of the interval $[a, b]$, $a = (1-\sqrt{y})^2$, $b = (1+\sqrt{y})^2$. We shall study the rate of convergence $\mathcal{F}_n(x)$ to the Marchenko–Pastur law assuming that $\Pr\{|X_{jk}| > t\} \le \varkappa^{-1}\exp\{-t^{\varkappa}\}$ for some $\varkappa > 0$. This problem has been studied by several authors. In particular, we proved in [10] that the Kolmogorov distance between $\mathcal{F}_n(x)$ and the distribution function $G_y(x)$, $\Delta_n^* := \sup_x |\mathcal{F}_n(x) - G_y(x)|$ is of order $O_P(n^{-\frac12})$. Bai et al. showed in [1] that $\Delta_n := \sup_x |F_n(x) - G_y(x)| = O(n^{-\frac12})$. For the Laguerre Unitary Ensemble we proved in [4] that $\Delta_n = O(n^{-1})$. Let $y = \frac{n}{p} \in (0, 1]$ in the what follows. For any positive constants $\alpha > 0$ and $\varkappa > 0$ define the quantities

$$l_{n,\alpha} := \log n\,(\log\log n)^{\alpha} \quad \text{and} \quad \beta_n := (l_{n,\alpha})^{\frac{1}{\varkappa}+\frac12}. \tag{1.1}$$

The main result of this paper is the following

Theorem 1.1. *Let $\mathbf{E}X_{jk} = 0$, $\mathbf{E}X_{jk}^2 = 1$ and there exists a constant $\varkappa > 0$ such that for any $1 \le j \le n$ and $1 \le k \le p$ and any $x \ge 1$,*

$$\Pr\{|X_{jk}| \ge x\} \le \varkappa^{-1}\exp\{-x^{\varkappa}\}. \tag{1.2}$$

Then for any $\alpha > 0$ there exist a positive constants C and c, depending on $\varkappa$ and α such that

$$\Pr\{\sup_x |\mathcal{F}_n(x) - G_y(x)| > n^{-1}\beta_n^2\} \le C\exp\{-c l_{n,\alpha}\}. \tag{1.3}$$

We apply the result of Theorem 1.1 to investigation of eigenvectors of the matrix $\mathbf{W}$. Let $\mathbf{u}_j = (u_{j1}, \ldots, u_{jn})^T$ be eigenvector of the matrix $\mathbf{W}$ corresponding to eigenvalue s_j^2, $j = 1, \ldots, n$. We prove the following result.

Theorem 1.2. *Under condition of Theorem 1.1 for any $\alpha > 0$ there exist constants C, c, depending on $\varkappa$ and α such that*

$$\Pr\{\max_{1\le j,k\le n} |u_{jk}|^2 > \frac{\beta_n^2}{n}\} \le C\exp\{-c l_{n,\alpha}\} \tag{1.4}$$

and

$$\Pr\{\max_{1\le k\le n} |\sum_{\nu=1}^{k} |u_{j\nu}|^2 - \frac{k}{n}| > \frac{\beta_n}{\sqrt{n}}\} \le C\exp\{-c l_{n,\alpha}\}. \tag{1.5}$$

We use a relatively short recursion argument based on the approach developed in [5] and [6] and ideas similar to those used in Erdös, Yau and Yin [8], Lemma 3.4.

2 Proof of the main Theorem

To bound Δ_n^* we shall use an approach developed in Götze and Tikhomirov, [10] and [5]. We shall apply a bound of the Kolmogorov distance between distribution functions via distance using their Stieltjes transforms. We denote the Stieltjes transform of $\mathcal{F}_n(x)$ by $m_n(z)$ and Stieltjes transform of the Marchenko–Pastur law by $s_y(z)$. We shall use the "symmetrization" of the spectrum sample covariance matrix as in [5]. Introduce the $(p+n)\times(p+n)$ matrix

$$\mathbf{V}=\frac{1}{\sqrt{p}}\begin{bmatrix}\mathbf{O} & \mathbf{X}\\ \mathbf{X}^* & \mathbf{O}\end{bmatrix}, \tag{2.1}$$

where $\mathbf{O}$ denotes a matrix with zero entries. Note that the eigenvalues of the matrix $\mathbf{V}$ are $\pm s_1,\dots,\pm s_n$, and 0 with multiplicity $p-n$. Let $\mathbf{R}=\mathbf{R}(z)$ denote the resolvent matrix of $\mathbf{V}$ defined by the equality

$$\mathbf{R}=(\mathbf{V}-z\mathbf{I}_{n+p})^{-1},$$

for all $z=u+iv$ with $v\neq 0$. Here and in what follows $\mathbf{I}_k$ denotes the identity matrix of order k. Sometimes we shall omit the sub index in the notation of the identity matrix. It is well-known that the Stieltjes transform of the Marchenko–Pastur distribution satisfies the equation

$$y\,z\,s^2(z)+(y-1+z)\,s_y(z)+1=0$$

(see, for example, equality (3.9) in [4]). If we consider the Stieltjes transforms $\widehat{s}_y(z)$ of the "symmetrized" Marchenko-Pastur distribution $\widehat{G}_y(x)=\frac{1+\operatorname{sign}x\,G_y(x^2)}{2}$, then it is straightforward to check that $\widehat{s}_y(z)=z\,s_y(z^2)$ and

$$y\widehat{s}^2(z)+(\frac{y-1}{z}+z)\widehat{s}_y(z)+1=0. \tag{2.2}$$

(see Section 3 in [5]). Furthermore, for the Stieltjes transform of the "symmetrized" empirical spectral distribution function $\widehat{\mathcal{F}}_n(x)=\frac{1+\operatorname{sign}x\,\mathcal{F}_n(x^2)}{2}$ we have

$$m_n(z)=\frac{1}{n}\sum_{j=1}^{n}R_{jj}=\frac{1}{n}\sum_{j=n+1}^{n+p}R_{jj}+\frac{1-y}{yz}.$$

(see, for instance, Section 3 in [5]). In the what follows we shall consider symmetrized random values only. We shall omit the symbol $\widehat{\cdot}$ in the notation of the distribution function and its Stieltjes transform. Let $\mathbb{T}_j=\{1,\dots,n\}\setminus\{j\}$. For $j=1,\dots,n$, introduce the matrices $\mathbf{V}^{(j)}$, obtained from $\mathbf{V}$ by deleting the j-th row and j-th column, and define the corresponding resolvent matrix $\mathbf{R}^{(j)}$ by the equality $\mathbf{R}^{(j)}=(\mathbf{V}^{(j)}-zI_{n+p-1})^{-1}$. Let $m_n^{(j)}(z)=\frac{1}{n}\sum_{l\in\mathbb{T}_j}R_{ll}^{(j)}$.

We shall use the representation, for $j = 1, \dots, n$,

$$R_{jj} = \frac{1}{-z - \frac{1}{p}\sum_{k,l=1}^{p} X_{jk}X_{jl}R^{(j)}_{k+n,l+n}} \tag{2.3}$$

(see, for example, Section 3 in [5]). We may rewrite it as follows

$$R_{jj} = -\frac{1}{z + ym_n(z) + \frac{y-1}{z}} + \frac{1}{z + ym_n(z) + \frac{y-1}{z}}\varepsilon_j R_{jj}, \tag{2.4}$$

where $\varepsilon_j = \varepsilon_{j1} + \varepsilon_{j2} + \varepsilon_{j3}$ and

$$\varepsilon_{j1} := \frac{1}{p}\sum_{k=1}^{p}(X_{jk}^2 - 1)R^{(j)}_{k+n,k+n}, \quad \varepsilon_{j2} := \frac{1}{p}\sum_{1 \le k \ne l \le p} X_{jk}X_{jl}R^{(j)}_{k+n,l+n},$$
$$\varepsilon_{j3} := \frac{1}{p}\Big(\sum_{l=1}^{p} R^{(j)}_{l+n,l+n} - \sum_{l=1}^{p} R_{l+n,l+n}\Big).$$

This relation immediately implies the following two equations

$$\begin{aligned} R_{jj} = &-\frac{1}{z + ym_n(z) + \frac{y-1}{z}} - \sum_{\nu=1}^{2}\frac{\varepsilon_{j\nu}}{(z + ym_n(z) + \frac{y-1}{z})^2} + \\ &\sum_{\nu=1}^{2}\frac{1}{(z + ym_n(z) + \frac{y-1}{z})^2}\varepsilon_{j\nu}\varepsilon_j R_{jj} + \frac{1}{z + ym_n(z) + \frac{y-1}{z}}\varepsilon_{j3}R_{jj}, \end{aligned} \tag{2.5}$$

and

$$\begin{aligned} m_n(z) &= -\frac{1}{z + ym_n(z) + \frac{y-1}{z}} + \frac{1}{(z + ym_n(z) + \frac{y-1}{z})}\frac{1}{n}\sum_{j=1}^{n}\varepsilon_j R_{jj} \\ &= -\frac{1}{z + ym_n(z) + \frac{y-1}{z}} - \frac{1}{(z + ym_n(z) + \frac{y-1}{z})^2}\frac{1}{n}\sum_{\nu=1}^{2}\sum_{j=1}^{n}\varepsilon_{j\nu} \\ &+ \frac{1}{(z + ym_n(z))^2}\frac{1}{n}\sum_{\nu=1}^{2}\sum_{j=1}^{n}\varepsilon_{j\nu}\varepsilon_j R_{jj} + \frac{1}{z + ym_n(z) + \frac{y-1}{z}}\frac{1}{n}\sum_{j=1}^{n}\varepsilon_{j3}R_{jj}. \end{aligned} \tag{2.6}$$

2.1 Large deviations I

In the following Lemmas we bound $\varepsilon_{j\nu}$, for $\nu = 1, 2, 3$ and $j = 1, \dots, n$.

Lemma 2.1. *Under the conditions of Theorem 1.1 we have, for any $z = u + iv$ with $v > 0$ and any $j = 1, \dots, n$,*

$$|\varepsilon_{j3}| \le \frac{1}{nv}.$$

Proof. It is straightforward to check that

$$\sum_{l=1}^{p} R_{l+n,l+n} = nm_n(z) - \frac{p-n}{z} = \sum_{l=1}^{p+n} R_{ll} - nm_n(z) \tag{2.7}$$

and

$$\sum_{l=1}^{p} R^{(j)}_{l+n,l+n} = \sum_{l=1,l\neq j}^{p+n} R^{(j)}_{ll} - (n-1)m^{(j)}{}_n(z) \tag{2.8}$$

Furthermore,

$$nm_n(z) = \frac{1}{2}\mathrm{Tr}\,\mathbf{R} + \frac{p-n}{2z}, \quad (n-1)m^{(j)}{}_n(z) = \frac{1}{2}\mathrm{Tr}\,\mathbf{R}^{(j)} + \frac{p-n+1}{2z}. \tag{2.9}$$

This implies

$$\sum_{l=1}^{p} R_{l+n,l+n} - \sum_{l=1}^{p} R^{(j)}_{l+n,l+n} = \frac{1}{2}\mathrm{Tr}\,\mathbf{R} - \frac{1}{2}\mathrm{Tr}\,\mathbf{R}^{(j)} - \frac{1}{z}. \tag{2.10}$$

The conclusion of Lemma 2.1 follows immediately from the inequality $|\mathrm{Tr}\,\mathbf{R} - \mathrm{Tr}\,\mathbf{R}^{(j)}| \le v^{-1}$ and $|\frac{1}{z}| \le v^{-1}$ (see Lemma 4.1 in [3]). □

Lemma 2.2. *Assuming conditions of Theorem 1.1, for any $\alpha > 0$ there exist positive constants C and c, depending on α and $\varkappa$ only such that for any $v > 0$, the following inequality holds*

$$\Pr\{|\varepsilon_{j1}| > 2n^{-\frac{1}{2}} l_{n,\alpha}^{\frac{2}{\varkappa}+\frac{1}{2}} (n^{-1}\sum_{l=1}^{p} |R^{(j)}_{l+n,l+n}|^2)^{\frac{1}{2}}\} \le C\exp\{-cl_{n,\alpha}\}$$

Proof. The proof of this Lemma is similar to the proof of Lemma 2.3 in [6]. We use the following inequality for the sum of independent random variables. Let $\xi_1, \ldots, \xi_p$ be independent random variables such that $\mathbf{E}\xi_j = 0$ and $|\xi_j| \le \sigma_j$. Then

$$\Pr\{|\sum_{j=1}^{p} \xi_j| > x\} \le c(1 - \Phi(x/\sigma) \le \frac{\sigma}{x}\exp\{-\frac{x^2}{2\sigma^2}\}, \tag{2.11}$$

where $\Phi(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} \exp\{-\frac{y^2}{2}\}dy$ and $\sigma^2 = \sigma_1^2 + \cdots + \sigma_p^2$. We put, for $k = 1, \ldots, p$, $\eta_k = X_{jk}^2 - 1$, and define

$$\xi_k = (\eta_k \mathbb{I}\{|X_{jk}| \le l_{n,\alpha}^{\frac{1}{\varkappa}})\} - \mathbf{E}\eta_k \mathbb{I}\{|X_{jk}| \le l_{n,\alpha}^{\frac{1}{\varkappa}}\})R^{(j)}_{kk}.$$

Note that $\mathbf{E}\xi_k = 0$ and $|\xi_k| \le 2l_{n,\alpha}^{\frac{2}{\varkappa}}|R^{(j)}_{k+n,k+n}|$. For any $j = 1, \ldots, n$, introduce the σ-algebra $\mathfrak{M}^{(j)}$ generated by the random variables X_{lk} with $1 \le l \ne j \le n, 1 \le k \le p$. Let $\mathbf{E}_j$ and $\Pr_j$ denote the conditional expectation and probability given $\mathfrak{M}^{(j)}$. Note

that random variables X_{jk} and σ-algebra $\mathfrak{M}^{(j)}$ are independent. Applying inequality (2.11) with $x := 2n^{\frac12} l_{n,\alpha}^{\frac{2}{\varkappa}+\frac12}\Big(n^{-1}\sum_{k=1}^p |R^{(j)}_{k+n,k+n}|^2\Big)^{\frac12}$, we get

$$\Pr\{|\sum_{k=1}^p \xi_k| > x\} = \mathbf{E}\Pr_j\{|\sum_{k=1}^p \xi_k| > x\} \le \mathbf{E}\exp\{-\frac{x^2}{\sigma^2}\} \le C\exp\{-cl_{n,\alpha}\}. \tag{2.12}$$

Furthermore, note that, for $k = 1, \dots, p$,

$$|\mathbf{E}_j\eta_k\mathbb{I}\{|\xi_k| \le l_{n,\alpha}^{\frac{2}{\varkappa}}\}| \le \mathbf{E}_j^{\frac12}|\eta_k|^2\Pr_j^{\frac12}\{|\xi_k| > l_{n,\alpha}^{\frac{1}{\varkappa}}\} \le \mathbf{E}^{\frac12}|\eta_k|^2\exp\{-\frac{c}{2}l_{n,\alpha}\}. \tag{2.13}$$

The last inequality implies that

$$|\sum_{k=1}^p \mathbf{E}_j\eta_k\mathbb{I}\{|X_{jk}| \le l_{n,\alpha}^{\frac{1}{\varkappa}}\})R^{(j)}_{k+n,k+n}| \le C\exp\{-\frac{c}{2}l_{n,\alpha}\}(\frac1n\sum_{l\in\mathbb{T}_j}|R^{(j)}_{k+n,k+n}|^2)^{\frac12}. \tag{2.14}$$

The inequalities (2.12) and (2.14) together conclude the proof of Lemma 2.2. Thus the Lemma is proved.

□

Corollary 2.3. *Under the conditions of Theorem 1.1 there exist constants c and C depending on α and $\varkappa$ only such that for any $z = u + iv$ with $v > 0$,*

$$\Pr\Big\{|\varepsilon_{j1}| > \beta_n^2(nv)^{-\frac12}\Big(\operatorname{Im} m_n^{(j)}(z) + \frac{(1-y)v}{|z|^2}\Big)^{\frac12}\Big\} \le C\exp\{-cl_{n,\alpha}\}. \tag{2.15}$$

Proof. Note that

$$\frac1n\sum_{k=1}^p |R^{(j)}_{k+n,k+n}|^2 \le \frac1n \operatorname{Tr}|\mathbf{R}^{(j)}|^2 = \frac{1}{nv}\operatorname{Im}\operatorname{Tr}\mathbf{R}^{(j)} \le \frac{2}{v}\operatorname{Im} m_n^{(j)}(z) + \frac{p-n+1}{nv}\operatorname{Im}\frac1z, \tag{2.16}$$

where $|\mathbf{R}^{(j)}|^2 = \mathbf{R}^{(j)}\mathbf{R}^{(j)*}$. The result follows now from Lemma 2.11. □

Lemma 2.4. *Under the conditions of Theorem 1.1, for any $j = 1, \dots, n$ and for any $v > 0$, the following inequality holds*

$$\Pr\Big\{|\varepsilon_{j2}| > \beta_n^2 n^{-\frac12}\Big(\frac1n\sum_{1\le k\ne l\le p}|R^{(j)}_{k+n,l+n}|^2\Big)^{\frac12}\Big\} \le C\exp\{-cl_{n,\alpha}\}. \tag{2.17}$$

Proof. We shall use a bound for the large deviation probabilities of quadratic forms which follows from results of Ledoux (see [9]).

Proposition 2.1. *Let $\xi_1, \dots, \xi_p$ be independent random variables such that $|\xi_k| \le 1$. Let also a_{lk} be complex numbers such that $a_{lk} = a_{kl}$ and $a_{kk} = 0$. Let $Z = \sum_{l,k=1}^p \xi_l\xi_k a_{lk}$. Let $\sigma^2 = \sum_{l,k=1}^p |a_{lk}|^2$. Then for every $t > 0$ there exists some positive constant $c > 0$ such that the following inequality holds*

$$\Pr\big\{|Z| \ge \frac32\mathbf{E}^{\frac12}|Z|^2 + t\big\} \le \exp\{-\frac{ct}{\sigma}\} \tag{2.18}$$

Proof. The result 2.1 follows from Theorem 3.1 [9]. $\square$

In order to bound ε_{j2} we use Proposition 2.1 with

$$\xi_k = (X_{jk}\mathbb{I}\{|X_{jk}| \le l_{n,\alpha}^{\frac{1}{\varkappa}}\} - \mathbf{E}X_{jk}\mathbb{I}\{|X_{jk}| \le l_{n,\alpha}^{\frac{1}{\varkappa}}\})/2l_{n,\alpha}^{\frac{1}{\varkappa}}, \tag{2.19}$$

for $k = 1, \dots, p$. Note that the random variables X_{jk}, $k = 1, \dots, p$ and the matrix $\mathbf{R}^{(j)}$ are mutually independent for any fixed $j = 1, \dots, n$. Moreover, $|\xi_k| \le 1$. Put $Z := \sum_{1\le k\ne l\le p} \xi_l\xi_k R_{kl}^{(j)}$. Applying Proposition 2.1 with $t := n^{\frac{1}{2}} l_{n,\alpha}(n^{-1}\sum_{1\le l\ne k\le p} |R_{lk}^{(j)}|^2)^{\frac{1}{2}}$, we get

$$\mathbf{E}\Pr{}_j\{|Z| \ge n^{\frac{1}{2}} l_{n,\alpha}(n^{-1}\sum_{1\le l\ne k\le p} |R_{lk}^{(j)}|^2)^{\frac{1}{2}}\} \le C\exp\{-cl_{n,\alpha}\}. \tag{2.20}$$

Furthermore,

$$\Pr\{\exists j \in [1,\dots,n] \,\text{and}\, \exists k \in [1,\dots,p] : |X_{jk}| > l_{n,\alpha}^{\frac{1}{\varkappa}}\} \le C\exp\{-cl_{n,\alpha}\} \tag{2.21}$$

and, for any $k = 1, \dots, p$,

$$\begin{aligned} |\mathbf{E}X_{jk}\mathbb{I}\{|X_{jk}| \le l_{n,\alpha}^{\frac{1}{\varkappa}}\}| &\le \Pr{}^{\frac{1}{2}}\{\exists j \in [1,\dots,n],\, k \in [1,\dots,p] : |X_{jk}| > l_{n,\alpha}^{\frac{1}{\varkappa}}\} \\ &\le C\exp\{-cl_{n,\alpha}\}. \end{aligned} \tag{2.22}$$

Introduce the random variables

$$\widehat{\xi}_k := X_{jk}\mathbb{I}\{|X_{jk}| \le l_{n,\alpha}^{\frac{1}{\varkappa}}\}/2l_{n,\alpha}^{\frac{1}{\varkappa}} \quad \text{and} \quad \widehat{Z} := \sum_{l,k=1}^{p} \widehat{\xi}_l\widehat{\xi}_k R_{l+n,k+n}^{(j)}.$$

Note that

$$\Pr\Big\{\sum_{l,k=1}^{p} X_{jk}X_{jl}R_{k+n,l+n}^{(j)} \ne \widehat{Z}\Big\} \le C\exp\{-cl_{n,\alpha}\}. \tag{2.23}$$

Inequalities (2.20)–(2.23) together imply

$$\Pr\Big\{|\varepsilon_{j2}| > l_{n,\alpha}^{\frac{2}{\varkappa}+1} n^{-\frac{1}{2}}(\frac{1}{n}\sum_{1\le k\ne l\le p} |R_{k+n,l+n}^{(j)}|^2)^{\frac{1}{2}}\Big\} \le C\exp\{-cl_{n,\alpha}\}. \tag{2.24}$$

Thus, Lemma 2.4 is proved. $\square$

Corollary 2.5. *Under the conditions of Theorem 1.1 there exist constants c and C, depending on $\varkappa$ and α such that for any $z = u + iv$ with $v > 0$*

$$\Pr\Big\{|\varepsilon_{j2}| > \beta_n^2 (nv)^{-\frac{1}{2}}\Big(\operatorname{Im} m_n^{(j)}(z) + \frac{(p-n+1)v}{n|z|^2}\Big)^{\frac{1}{2}}\Big\} \le C\exp\{-cl_{n,\alpha}\}. \tag{2.25}$$

Proof. Note that

$$n^{-1}\sum_{1\le k\ne l\le p}|R^{(j)}_{k+n,l+n}|^2\le n^{-1}\mathrm{Tr}\,|\mathbf{R}^{(j)}|^2\le\frac{2}{v}\mathrm{Im}\,m_n^{(j)}(z)+\frac{(p-n+1)v}{n|z|^2}. \tag{2.26}$$

The result follows now from Lemma 2.4.

□

Collecting these results recall the definition

$$\beta_n=l_{n,\alpha}^{\frac{1}{\varkappa}+\frac12}. \tag{2.27}$$

Then we may write, for $\nu=1,2,3$

$$\Pr\Big\{|\varepsilon_{j\nu}|>\frac{\beta_n}{\sqrt{n}}\Big(\frac{(\mathrm{Im}\,m_n(z))^{\frac12}}{\sqrt{v}}+\frac{\sqrt{1-y}}{|z|}+\frac{1}{\sqrt{nv}}\Big)\Big\}\le C\exp\{-cl_{n,\alpha}\}. \tag{2.28}$$

Denote by

$$\Omega_n(z):=\Big\{\omega\in\Omega:\,|\varepsilon_j|\le\frac{2\beta_n}{\sqrt{n}}\Big(\frac{(\mathrm{Im}\,m_n(z)m)^{\frac12}}{\sqrt{v}}+\frac{1}{\sqrt{nv}}+\frac{\sqrt{1-y}}{|z|}\Big)\Big\}. \tag{2.29}$$

Let $v_0:=\frac{d\beta_n}{n}$ with sufficiently small positive constant d. We introduce the region $\mathcal{D}:=\{z=u+iv\in\mathbb{C}:|u|\in\mathcal{J}_\varepsilon,\,v_0<v\le2\}$ and a sequence $z_l=u_l+v_l$ in $\mathcal{D}$, defined recursively via $u_{l+1}-u_l=\frac{4}{n^8}$ and $v_{l+1}-v_l=\frac{2}{n^8}$. We introduce the events

$$\Omega'_n(z_l):=\Big\{\omega\in\Omega:\,|\varepsilon_j|\le\frac{\beta_n}{\sqrt{n}}\Big(\frac{(\mathrm{Im}\,m_n(z))^{\frac12}}{\sqrt{v}}+\frac{1}{\sqrt{nv}}+\frac{\sqrt{1-y}}{|z|}\Big)\Big\}. \tag{2.30}$$

Using a union bound we obtain

$$\Pr\{\cap_{z_l\in\mathcal{D}}\Omega'_n(z_l)\}\ge1-C\exp\{-cl_{n,\alpha}\}. \tag{2.31}$$

It is straightforward to check that

$$|\varepsilon_j(z)-\varepsilon_j(z')|\le\frac{|z-z'|}{v_0^2}\frac{\beta_n}{\sqrt{n}}\Big(\frac{(\mathrm{Im}\,m_n(z))^{\frac12}}{\sqrt{v}}+\frac{1}{\sqrt{nv}}+\frac{\sqrt{1-y}}{|z|}\Big) \tag{2.32}$$

This immediately implies that, for $|z-z_l|\le\frac{5}{n^8}$,

$$\Omega_n(z)\subset\Omega'_n(z_l) \tag{2.33}$$

and

$$\Pr\{\cap_{z\in\mathcal{D}}\Omega_n(z)\}\ge1-C\exp\{-cl_{n,\alpha}\} \tag{2.34}$$

with some constants C and c, depending on α and $\varkappa$ only.

3 Large deviations II

In this Section we obtain bounds for the large deviation probabilities of the sum of the ε_j. We start with the quantity

$$\delta_{n1}=\frac{1}{n^2}\sum_{j=1}^{n}\sum_{k=1}^{p}(X_{jl}^2-1)R^{(j)}_{k+n,k+n}. \tag{3.1}$$

We prove the following Lemma

Lemma 3.1. *Assume that there exists a constant C_0 such that for any $j=1,\dots,n$ and any $k=1,\dots,p$*

$$\operatorname{Im} m_n^{(j)}(z)\le C_0. \tag{3.2}$$

Then there exist constants c and C, depending on α, $\varkappa$ and C_0 such that

$$\Pr\{|\delta_{n1}|\le n^{-1}v^{-\frac12}\beta_n^2\}\le C\exp\{-cl_{n,\alpha}\} \tag{3.3}$$

Proof. For any $j=1,\dots,n$ and any $k=1,\dots,p$, we introduce the truncated random variables

$$\xi_{jk}=\widehat{X}_{jk}^2-\mathbf{E}\widehat{X}_{jk}^2, \tag{3.4}$$

where $\widehat{X}_{jk}=X_{jk}\mathbb{I}\{|X_{jk}|\le l_{n,\alpha}^{\frac{1}{\varkappa}}\}$. It is straightforward to check that

$$0\le 1-E\widehat{X}_{jk}^2\le C\exp\{-cl_{n,\alpha}\}. \tag{3.5}$$

Introduce as well the quantities

$$\widehat{\delta}_{n1}=\frac{1}{n^2}\sum_{j=1}^{n}\sum_{k=1}^{p}(\widehat{X}_{jk}^2-1)R^{(j)}_{k+n,k+n},\quad \widetilde{\delta}_{n1}=\frac{1}{n^2}\sum_{j=1}^{n}\sum_{k=1}^{p}\xi_{lk}R^{(j)}_{k+n,k+n} \tag{3.6}$$

By assumption (1.2),

$$\Pr\{\delta_{n1}\ne\widehat{\delta}_{n1}\}\le C\exp\{-cl_{n,\alpha}\}. \tag{3.7}$$

Let

$$\zeta_j:=\frac{1}{\sqrt{n}}\sum_{k=1}^{p}\xi_{jk}R^{(j)}_{k+n,k+n}. \tag{3.8}$$

Then

$$\widehat{\delta}_{n1}=\frac{1}{n^2}\sum_{j=1}^{n}\zeta_j. \tag{3.9}$$

Note that the sequence $\widehat{\delta}_{n1}$ is a martingale with respect to the σ–algebras $\mathfrak{M}_j$. In fact,

$$\mathbf{E}\big(\zeta_j|\mathfrak{M}_{j-1}\big)=\mathbf{E}\Big(\mathbf{E}\big(\zeta_j|\mathfrak{M}^{(j)}\big)|\mathfrak{M}_{j-1}\Big)=0. \tag{3.10}$$

In order to use large deviation bounds for $\widehat{\delta}_{n1}$ we replace the differences ζ_j by truncated random variables. Introduce

$$\widehat{\zeta}_j := \zeta_j \mathbb{I}\Big\{|\zeta_j| \le l_{n,\alpha}^{\frac{2}{\varkappa}+\frac{1}{2}}\Big(\frac{1}{n}\sum_{k=1}^{p}|R^{(j)}_{k+p,k+p}|^2\Big)^{\frac{1}{2}}\Big\}. \tag{3.11}$$

Since ζ_j is sum of independent bounded random variables with mean zero, we have

$$\Pr\Big\{|\zeta_j| > l_{n,\alpha}^{\frac{2}{\varkappa}+\frac{1}{2}}\Big(\frac{1}{n}\sum_{k=1}^{p}|R^{(j)}_{k+n,k+n}|^2\Big)^{\frac{1}{2}}\}\Big\} \le C\exp\{-cl_{n,\alpha}\}. \tag{3.12}$$

This implies that

$$\Pr\Big\{\sum_{j=1}^{n}\zeta_j \ne \sum_{j=1}^{n}\widehat{\zeta}_j\Big\} \le C\exp\{-cl_{n,\alpha}\}. \tag{3.13}$$

Furthermore, introduce the conditionally recentered random variables

$$\widetilde{\zeta}_j = \widehat{\zeta}_j - \mathbf{E}\{\widehat{\zeta}_j\Big|\mathfrak{M}_{j-1}\}. \tag{3.14}$$

Using the Cauchy-Schwarz inequality and boundedness of the random variables $\xi_{jk}R^{(j)}_{k+n,k+n}$ it follows that

$$|\mathbf{E}\{\widehat{\zeta}_j|\mathfrak{M}_{j-1}\}| \le C\exp\{-cl_{n,\alpha}\}. \tag{3.15}$$

Using a martingale bound by Bentkus, [2], Theorem 1.1, we obtain the following result. Let $\mathfrak{M}_0 = \{\emptyset, \Omega\} \subset \mathfrak{M}_1 \subset \cdots \subset \mathfrak{M}_n \subset \mathfrak{M}$ be a family of sub-σ-algebras of the measurable space $\{\Omega, \mathfrak{M}\}$ and let $M_n = \xi_1 + \ldots + \xi_n$ be a martingale with bounded differences $\xi_j = M_j - M_{j-1}$ such that

$$\Pr\{|\xi_j| \le b_j\} = 1 \quad \text{for} \quad j = 1, \ldots, n.$$

Then, for $x > \sqrt{8}$,

$$\Pr\Big\{|M_n| \ge x\Big\} \le c(1 - \Phi(\frac{x}{\sigma})) = \int_{\frac{x}{\sigma}}^{\infty}\varphi(t)dt, \quad \varphi(t) = \frac{1}{\sqrt{2\pi}}\exp\{\frac{-t^2}{2}\}. \tag{3.16}$$

with some numerical constant $c > 0$ and $\sigma^2 = b_1^2 + \cdots + b_n^2$. Note that for $t > C$

$$1 - \Phi(t) \le \frac{1}{C}\varphi(t).$$

We shall use now the inequality

$$\Pr\Big\{|M_n| \ge x\Big\} \le \exp\{-\frac{x^2}{2\sigma^2}\} \tag{3.17}$$

to bound $\widetilde{\delta}_{n1}$. By assumption (3.2) and the definition of $\widetilde{\zeta}_j$, we may choose $\beta_j = l_{n,\alpha}^{\frac{2}{\varkappa}+\frac{1}{2}}$, obtaining

$$\Pr\{|\widetilde{\delta}_{n1}| > n^{-1}v^{-\frac{1}{2}}\mathrm{Im}^{\frac{1}{2}}m_n^{(j)}(z)l_{n,\alpha}^{\frac{2}{\varkappa}+1}\} \le C\exp\{-cl_{n,\alpha}\}. \tag{3.18}$$

Inequalities (3.13)–(3.18) together conclude the proof of Lemma 3.1. □

Let

$$\delta_{n2} := \frac{1}{n^2}\sum_{j=1}^{n}\sum_{1\le l\ne k\le p} X_{jl}X_{jk}R^{(j)}_{l+n,k+n}. \tag{3.19}$$

Lemma 3.2. *Assume that there exists a constant C_0 such that for any $j=1,\dots n$*

$$\operatorname{Im} m_n^{(j)}(z) \le C_0. \tag{3.20}$$

Then there exist constants c and C, depending on α, $\varkappa$ and C_0 such that

$$\Pr\{|\delta_{n2}| > \frac{1}{n\sqrt{v}}\beta_n^2 \operatorname{Im}^{\frac12} m_n^{(j)}(z)\} \le C\exp\{-c l_{n,\alpha}\} \tag{3.21}$$

Proof. The proof of this Lemma is similar to proof of Lemma 3.1. We introduce the random variables

$$\eta_j = \frac{1}{\sqrt{n}}\sum_{1\le l\ne k\le p} X_{jk}X_{jl}R^{(j)}_{l+n,k+n} \tag{3.22}$$

and note that the sequence

$$M_n = \sum_{j=1}^{n}\eta_j \tag{3.23}$$

is a martingale with respect to the σ–algebras $\mathfrak{M}_j$. We now apply the martingale large bound (3.17) twice, replacing η_j by truncated random variables. Thus Lemma is proved. □

Finally we have to bound

$$\delta_{n3} := \frac{1}{n^2}\sum_{j=1}^{n}\sum_{k=1}^{p}(R_{k+n,k+n} - R^{(j)}_{k+n,k+n})R_{jj}. \tag{3.24}$$

Lemma 3.3. *There exists a positive constant C such that*

$$|\delta_{n3}| \le \frac{1}{nv}\operatorname{Im} m_n(z). \tag{3.25}$$

Proof. It is easy to check that

$$\sum_{k=1}^{p}(R_{k+n,k+n} - R^{(j)}_{k+n,k+n}) = \frac{1}{2}(\operatorname{Tr}\mathbf{R} - \operatorname{Tr}\mathbf{R}^{(j)}) + \frac{1}{2z} \tag{3.26}$$

By formula (5.4) in [3], we have

$$(\operatorname{Tr}\mathbf{R} - \operatorname{Tr}\mathbf{R}^{(j)})R_{jj} = (1 + \frac{1}{p}\sum_{l,k=1}^{n} X_{jl}X_{jk}(R^{(j)})^2)_{l+n,k+n})R_{jj}^2 = -\frac{d}{dz}R_{jj}. \tag{3.27}$$

From here it follows that

$$\frac{1}{n^2}\sum_{j=1}^{n}(\operatorname{Tr}\mathbf{R}-\operatorname{Tr}\mathbf{R}^{(j)})R_{jj}=-\frac{1}{n^2}\frac{d}{dz}m_n(z). \tag{3.28}$$

Note that

$$m_n(z)=\frac{z}{n}\sum_{k=1}^{n}\frac{1}{s_k^2-z^2} \tag{3.29}$$

and

$$\frac{d}{dz}m_n(z)=\frac{m_n(z)}{z}-\frac{2z^2}{n}\sum_{k=1}^{n}\frac{1}{(s_k^2-z^2)^2}. \tag{3.30}$$

This implies that

$$\delta_{n3}=\frac{z^2}{n^2}\sum_{k=1}^{n}\frac{1}{(s_k^2-z^2)^2}. \tag{3.31}$$

Finally, we note that

$$\Big|\frac{z^2}{n^2}\frac{1}{(s_k^2-z^2)^2}\Big|\le\frac{1}{nv}\operatorname{Im}m_n(z). \tag{3.32}$$

The last inequality concludes the proof. Thus Lemma 3.3 is proved. □

3.1 Stieltjes transforms

In this section we derive auxiliary bounds for the difference between the Stieltjes transforms $m_n(z)$ of the empirical spectral measure of the matrix $\mathbf{V}$ and the Stieltjes transform $s_y(z)$ of the symmetrized Marchenko–Pastur law. We introduce the additional notations

$$\delta_n=\delta_{n1}+\delta_{n2},\quad \widehat{\delta}_n=\delta_{n3},\quad \overline{\delta}_n=\frac{1}{n}\sum_{\nu=1}^{3}\sum_{j=1}^{n}\varepsilon_{j\nu}\varepsilon_jR_{jj}. \tag{3.33}$$

Recall that $s_y(z)$ satisfies the equation

$$s_y(z)=-\frac{1}{z+ys_y(z)+\frac{y-1}{z}}. \tag{3.34}$$

Denote by $g_n(z)=m_n(z)-s_y(z)$. The representation (2.6) and the equality (3.34) together imply

$$\begin{aligned}g_n(z)=&\frac{yg_n(z)s_y(z)}{z+ym_n(z)+\frac{y-1}{z}}-\frac{\delta_n}{(z+ym_n(z)+\frac{y-1}{z})^2}\\&+\frac{\widehat{\delta}_n}{z+ym_n(z)+\frac{y-1}{z}}+\frac{\overline{\delta}_n}{(z+ym_n(z)+\frac{y-1}{z})^2}.\end{aligned} \tag{3.35}$$

Introduce the following notations

$$b_n(z,y) = z + y m_n(z) + \frac{y-1}{z}, \quad \text{and} \quad a_n(z,y) = b_n(z,y) + y s_y(z).$$

From equality (3.35) it follows that

$$|g_n(z)| \le \frac{|\delta_n| + |\overline{\delta}_n|}{|b_n(z,y)||a_n(z,y)|} + \frac{|\widehat{\delta}_n|}{|a_n(z,y)|}. \tag{3.36}$$

For any $z \in \mathcal{D}$ introduce the events

$$\widehat{\Omega}_n(z) = \Big\{\omega \in \Omega:\ |\delta_n| \le \frac{\beta_n \mathrm{Im}^{\frac12} m_n^{(j)}(z)}{n\sqrt{v}}\Big\}, \quad \widetilde{\Omega}_n(z) = \Big\{\omega \in \Omega:\ |\widehat{\delta_n}| \le \frac{\mathrm{Im}\, m_n(z)}{nv}\Big\}, \tag{3.37}$$

$$\overline{\Omega}_n(z) = \Big\{\omega \in \Omega:\ |\overline{\delta}_n| \le (\frac{\beta_n^2 \mathrm{Im}\, m_n(z)}{nv} + \frac{\beta_n^2}{(nv)^2} + \frac{1-y}{n|z|^2})\frac{1}{n}\sum_{j=1}^n |R_{jj}|^2\Big\}. \tag{3.38}$$

Put $\Omega_n^*(z) = \widehat{\Omega}_n(z) \cup \widetilde{\Omega}_n(z) \cup \overline{\Omega}_n(z)$. By Lemmas 3.1–3.3, we have

$$\Pr\{\widehat{\Omega}_n(z)\} \ge 1 - C\exp\{-c l_{n,\alpha}\}. \tag{3.39}$$

By Lemma 3.3,

$$\Pr\{\widetilde{\Omega}_n(z)\} = 1. \tag{3.40}$$

Note that

$$|\varepsilon_{j\nu}\varepsilon_{j3}| \le \frac{1}{2}(|\varepsilon_{j\nu}|^2 + |\varepsilon_{j3}|^2). \tag{3.41}$$

By inequality (2.28), we have, for $\nu = 1, 2$,

$$\Pr\Big\{|\varepsilon_{j\nu}|^2 > \frac{\beta_n^2}{n}\Big(\frac{\mathrm{Im}\, m_n(z)}{v} + \frac{1}{nv^2} + \frac{(1-y)v}{|z|^2}\Big)\Big\} \le C\exp\{-c l_{n,\alpha}\}. \tag{3.42}$$

Furthermore,

$$\Pr\{|\varepsilon_{j3}|^2 \le \frac{1}{n^2v^2}\} = 1. \tag{3.43}$$

Similar to equality (2.34) we may show that

$$\Pr\Big\{\cap_{z\in\mathcal{D}}\big(\Omega_n^*(z) \cap \Omega_n(z)\big)\Big\} \ge 1 - C\exp\{-c l_{n,\alpha}\}. \tag{3.44}$$

Let

$$\Omega_n^* := \cap_{z\in\mathcal{D}}\big(\Omega_n^*(z) \cap \Omega_n(z)\big). \tag{3.45}$$

In the what follows we shall assume that

$$v_0 = \frac{b\beta_n}{n} \tag{3.46}$$

with $b \ge 32$. We now prove the first essential bound.

Lemma 3.4. *Let $z = u + iv \in \mathcal{D}$. Assume that*

$$|g_n(z)| \le \frac{1}{2}. \tag{3.47}$$

Then for any $\omega \in \Omega_n^*$, the following bound holds

$$|g_n(z)| \le \frac{4\beta_n^2}{nv} + \frac{32\beta_n^2}{n^2v^2\sqrt{\gamma+v}}.$$

Proof. First note that the inequality $|g_n(z)| \le \frac{1}{2}$ implies

$$|b_n(z,y)| = |z + ym_n(z) + \frac{y-1}{z}| \ge |z + ys_y(z) + \frac{y-1}{z}| - |yg_n(z)| \ge \frac{1}{2}. \tag{3.48}$$

Moreover, by definition of v_0 (3.46), we have

$$\operatorname{Im} m_n^{(j)}(z) \le |m_n^{(j)}(z)| \le |m_n(z)| + \frac{1}{nv} \le |s_y(z)| + |g_n(z)| + \frac{1}{nv} \le 2. \tag{3.49}$$

Furthermore, we have the obvious inequality

$$\begin{aligned} |a_n(z,y)| = |z + ym_n(z) + ys_y(z) + \frac{y-1}{z}| &\ge \operatorname{Im}(z + \frac{y-1}{z} + ys_y(z)) \\ &\ge \frac{1}{2}\operatorname{Im}\Big(\sqrt{(z+\frac{y-1}{z})^2 - 4y}\Big). \end{aligned} \tag{3.50}$$

Moreover,

$$|a_n(z,y)| \ge \operatorname{Im}\Big(\frac{y-1}{z}\Big) = \frac{v(1-y)}{|z|^2}. \tag{3.51}$$

For $z \in \mathcal{D}$ we obtain $\operatorname{Re}((z+\frac{y-1}{z})^2 - 4y) \le 0$ and $\frac{\pi}{2} \le \arg((z+\frac{y-1}{z})^2 - 4y) \le \frac{3\pi}{2}$. Therefore,

$$\operatorname{Im}\Big(\sqrt{(z+\frac{y-1}{z})^2 - 4y}\Big) \ge \frac{1}{\sqrt{2}}\Big|(z+\frac{y-1}{z})^2 - 4y\Big|^{\frac{1}{2}} \ge \frac{1}{4}\sqrt{\gamma+v}, \tag{3.52}$$

where $\gamma = \min(1+\sqrt{y}-|u|,\, 1-\sqrt{y}-|u|)$. Inequality (3.36) implies that for $\omega \in \Omega_n^*$

$$\begin{aligned} |g_n(z)| \le{}& \frac{\beta_n}{n\sqrt{v}|b_n(z,y)||a_n(z,y)|} + \frac{\operatorname{Im} m_n(z)}{nv|a_n(z,y)|} \\ &+ \frac{\beta_n^2}{nv|a_n(z,y)||b_n(z,y)|}\Big(\operatorname{Im} m_n(z) + \frac{1}{nv} + \frac{(1-y)v}{|z|^2}\Big)\frac{1}{n}\sum_{j=1}^{n}|R_{jj}|^2. \end{aligned} \tag{3.53}$$

Furthermore, equation (2.4), inequality (3.48) and definition of Ω_n^* in (2.30) together imply that, for $\omega \in \Omega_n^*$ and $z \in \mathcal{D}$

$$|R_{jj}| \le \frac{2}{|b_n(z,y)|} \le 4. \tag{3.54}$$

Inequalities (3.49)–(3.54) together imply

$$\begin{aligned}|g_n(z)| &\le \frac{2\beta_n}{n\sqrt{v}|a_n(z,y)|}\Big(1+\frac{4\beta_n \operatorname{Im} m_n(z)}{\sqrt{v}}+\frac{4\beta_n}{nv}\Big) \\ &\quad + \frac{2\beta_n^2(1-y)v}{nv|a_n(z,y)|}+\frac{\operatorname{Im} m_n(z)}{nv|a_n(z,y)|} \\ &\le \frac{4\beta_n^2}{nv}+\frac{32\beta_n^2}{n^2v^2\sqrt{\gamma+v}}. \end{aligned} \tag{3.55}$$

This inequality completes the proof of the Lemma. □

Put now $v_0' := v_0'(z) := \frac{v_0}{\sqrt{\gamma}}$, where $\gamma = \min\{|u|-1+\sqrt{y}, 1+\sqrt{y}-|u|$ and $z=u+iv$. Denote $\widehat{\mathcal{D}} := \{z\in\mathcal{D} :\ v\ge v_0'\}$.

Corollary 3.5. *Assume that for $\omega\in\Omega_n$ and $z=u+iv\in\widehat{\mathcal{D}}$,*

$$|g_n(z)| \le \frac{1}{2} \quad \textit{holds.}$$

Then

$$|g_n(z)| \le \frac{1}{4}.$$

Proof. Note that for $v\ge v_0'$

$$\frac{4\beta_n^2}{nv}+\frac{32\beta_n^2}{n^2v^2\sqrt{\gamma+v}} \le \frac{1}{4} \tag{3.56}$$

Thus the Corollary is proved. □

Corollary 3.6. *Let $z=u+iv\in\mathcal{D}$. Assume that*

$$|b_n(z,y)\ge\frac{1}{2}. \tag{3.57}$$

Then for any $\omega\in\Omega_n^*$, the following bound holds

$$|g_n(z)| \le \frac{4\beta_n^2}{nv}+\frac{32\beta_n^2}{n^2v^2\sqrt{\gamma+v}}.$$

Proof. The proof of Lemma 3.4 relies on the condition (3.47) of Lemma 3.4 only to prove inequality (3.57). Thus the Corollary is proved. □

The next lemma yields a recursion which plays a crucial role in our proof lemma 3.4 and is similar to an approach used in [7].

Lemma 3.7. *Assume that for some $z=u+iv\in\mathcal{D}$ with $v\ge v_0$ condition (3.47) holds. Then it holds for $z'=u+iv'\in\mathcal{D}$ such that $0<v-v'\le n^{-8}$ and $v'\ge v_0$ as well.*

Proof. First of all we note that

$$|m_n(z)-m_n(z')| = \frac{1}{n}(v-v')|\operatorname{Tr}\mathbf{R}(z)\mathbf{R}(z')| \le \frac{v-v'}{vv'} \le \frac{1}{b^2\beta_n^4 n^6} \le \frac{1}{8} \tag{3.58}$$

and

$$|s_y(z)-s_y(z')| \le \frac{1}{8} \tag{3.59}$$

By Corollary 3.6, we have

$$|g_n(z)| \le \frac{1}{4}. \tag{3.60}$$

All these inequalities together imply

$$|g_n(z')| \le \frac{1}{2}. \tag{3.61}$$

Thus the Lemma is proved. □

Proposition 3.1. *There exists constants C, c, such that*

$$\Pr\left\{|g_n(z)| > \frac{4\beta_n^2}{nv} + \frac{32\beta_n^2}{n^2v^2\sqrt{\gamma+v}}\right\} \le C\exp\{-cl_{n,\alpha}\}. \tag{3.62}$$

Proof. Note that for $v=2$ we have

$$|z+ym_n(z)+\frac{y-1}{z}| \ge \operatorname{Im} z \ge 2. \tag{3.63}$$

By Lemma 3.4, we obtain the inequality (3.62). By Lemma 3.7, this inequality holds for any $2 \ge v \ge v_0$. Thus proposition 3.1 is proved.

□

4 Proof of Theorem 1.1

To conclude the proof of Theorem 1.1 we modify a bound for the Kolmogorov distance between distribution functions based on their Stieltjes transforms obtained in [4], Lemma 2.1. Given $\varepsilon > 0$ introduce the interval $\mathbb{J}_\varepsilon = [1-\sqrt{y}+\varepsilon, 1+\sqrt{y}-\varepsilon]$ and $\mathbb{J}'_\varepsilon = [1-\sqrt{y}+\frac{1}{2}\varepsilon, 1+\sqrt{y}-\frac{1}{2}\varepsilon]$. For any x such that $|x| \in \mathbb{J}_\varepsilon$ define $\gamma = \gamma(x) := \min\{|x|-1+\sqrt{y}, 1+\sqrt{y}-|x|$. For a distribution function F denote by $S_F(z)$ its Stieltjes transform,

$$S_F(z) = \int_{-\infty}^{\infty} \frac{1}{x-z} dF(x).$$

Proposition 4.1. *Let $v > 0$ and a and $\varepsilon > 0$ be positive numbers such that*

$$\alpha = \frac{1}{\pi}\int_{|u|\le a} \frac{1}{u^2+1} du = \frac{3}{4}, \tag{4.1}$$

and

$$2va \leq \varepsilon\sqrt{\gamma}. \tag{4.2}$$

If G_y denotes the distribution function of the symmetrized Marchenko–Pastur law, and F is any distribution function, there exists some absolute constants C_1, C_2, C_3 depending on y only such that

$$\begin{aligned}\Delta(F, G_y) &:= \sup_x |F(x) - G_y(x)| \\ &\leq C_1 \sup_{x:|x|\in\mathbb{J}'_\varepsilon} \Big|\mathrm{Im} \int_{-\infty}^{x} (S_F(u + i\frac{v}{\sqrt{\gamma}}) - S_{G_y}(u + i\frac{v}{\sqrt{\gamma}}))du\Big| + C_2 v + C_3\varepsilon^{\frac{3}{2}}.\end{aligned} \tag{4.3}$$

Proof. Without loss of generality we may assume that $0 < y < 1$. The case $y = 1$ is considered in [6]. The proof of Proposition 4.1 is an adaption of the proof of Proposition 4.1 from [6]. We provide it here for completeness. Note that

$$\sup_x |F(x) - G_y(x)| \leq \sup_{x\in\mathbb{J}_\varepsilon} |F(x) - G_y(x)| + G_y(-1 - \sqrt{y} + \varepsilon), \tag{4.4}$$

and

$$G_y(-1 - \sqrt{y} + \varepsilon) \leq C\varepsilon^{\frac{3}{2}}. \tag{4.5}$$

Let $x \in \mathbb{J}_\varepsilon$. Recall that $\gamma = \min\{|x| - 1 + \sqrt{y}, 1 + \sqrt{y} - |x|\}$. Then, according to condition (4.2) we have $x + \frac{va}{\sqrt{\gamma}} \in \mathbb{J}'_\varepsilon$. Denote by $v' = \frac{v}{\sqrt{\gamma}}$. For any $x \in \mathbb{J}'_\varepsilon$ we have

$$\begin{aligned}&\Big|\frac{1}{\pi}\mathrm{Im}\Big(\int_{-\infty}^{x} (S_F(u + iv') - S_{G_y}(u + iv'))du\Big)\Big| \\ &\geq \frac{1}{\pi}\mathrm{Im}\Big(\int_{-\infty}^{x} (S_F(u + iv') - S_{G_y}(u + iv'))du\Big) \\ &= \frac{1}{\pi}\Big[\int_{-\infty}^{\infty} \frac{v'd(F(q) - G_y(q))}{(q-u)^2 + v'^2}\Big] du \\ &= \frac{1}{\pi}\int_{-\infty}^{x}\Big[\int_{-\infty}^{\infty} \frac{2v'(q-u)(F(q) - G_y(q))dq}{((q-u)^2 + v'^2)^2}\Big] \\ &= \frac{1}{\pi}\int_{-\infty}^{\infty}(F(q) - G_y(q))\Big[\int_{-\infty}^{x} \frac{2v'(q-u)du}{((q-u)^2 + v'^2)^2}dq\Big] \\ &= \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{(F(x - v'q) - G_y(x - v'q))dq}{q^2+1}.\end{aligned} \tag{4.6}$$

Since F is non decreasing, we obtain

$$\begin{aligned}&\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{(F(x - v'q) - G_y(x - v'q))dy}{q^2+1} \\ &\geq \alpha(F(x - v'a) - G_y(x - v'a)) - \frac{1}{\pi}\int_{|q|\leq a} |G_y(x - v'q) - G_y(x - v'a))|dq \\ &\geq \alpha(F(x - v'a) - G_y(x - v'a)) - \frac{1}{v'\pi}\int_{|q|\leq v'a} |G_y(x - q) - G_y(x - v'a))|dq.\end{aligned} \tag{4.7}$$

Denote by $\Delta_\varepsilon(F,G)=\sup_{x\in\mathbb{J}_\varepsilon}|F(x)-G_y(x)|$ and let $x_n\in\mathbb{J}_\varepsilon$ such that $F(x_n)-G_y(x_n)\to\Delta_\varepsilon(F,G)$. Then $x_n=x_n+v'a\in\mathbb{J}'_\varepsilon$. We have

$$\sup_{x\in\mathbb{J}'_\varepsilon}\left|\operatorname{Im}\int_{-\infty}^{x}(S_F(u+iv')-S_{G_y}(u+iv'))du\right|\geq\alpha(F(x_n)-G_y(x_n))$$
$$-\frac{1}{\pi v}\sup_{x\in\mathbb{J}'_\varepsilon}\frac{1}{\sqrt{\gamma}}\int_{|q|<2v'a}|G_y(x+q)-G_y(x)|dq-(1-\alpha)\Delta(F,G). \tag{4.8}$$

Note that

$$\frac{1}{\pi v}\sup_{x\in\mathbb{J}'_\varepsilon}\frac{1}{\sqrt{\gamma}}\int_{|q|<2v'a}|G_y(x+q)-G_y(x)|dy$$
$$\leq\frac{1}{\pi v}\sup_{x\in\mathbb{J}'_\varepsilon}\frac{1}{\sqrt{\gamma}}\sqrt{4-x^2}\leq Cv. \tag{4.9}$$

Inequalities (4.4), (4.8) and (4.9) together imply

$$\sup_{x\in\mathbb{J}'_\varepsilon}\left|\operatorname{Im}\int_{-\infty}^{x}(S_F(u+iv')-S_{G_y}(u+iv'))du\right|\geq(2\alpha-1)\Delta_\varepsilon(F,G)-Cv-C\varepsilon^{\frac{3}{2}}. \tag{4.10}$$

Similar arguments may be used for the sequence $x_n\in\mathbb{J}_\varepsilon$ such $F(x_n)-G_y(x_n)\to-\Delta_\varepsilon(F,G)$. This completes the proof. □

Corollary 4.1. *Under the conditions of Proposition 4.1, for any $V>v$, the following inequality holds*

$$\sup_{x\in\mathbb{J}'_\varepsilon}\left|\int_{-\infty}^{x}(\operatorname{Im}(S_F(u+iv')-S_{G_y}(u+iv'))du\right|$$
$$\leq\int_{-\infty}^{\infty}|S_F(u+iV)-S_{G_y}(u+iV)|du$$
$$+\sup_{x\in\mathbb{J}'_\varepsilon}\left|\int_{v'}^{V}\big(S_F(x+iu)-S_{G_y}(x+iu)\big)\,du\right|. \tag{4.11}$$

Proof. Set $z=u+iv'$. $v'\leq 2$. Since the functions of $S_F(z)$ and $S_{G_y}(z)$ are analytic in the upper half-plane, it is enough to use Cauchy's theorem. We can write

$$\int_{-\infty}^{\infty}\operatorname{Im}(S_F(z)-S_{G_y}(z))du=\lim_{L\to\infty}\int_{-L}^{x}(S_F(u+iv')-S_{G_y}(u+iv'))du. \tag{4.12}$$

Since $v'=\frac{v}{\sqrt{\gamma}}\leq\frac{\varepsilon}{2a}$, without loss of generality we may assume that $v'\leq 2$. By Cauchy's integral formula, we have

$$\int_{-L}^{x}(S_F(z)-S_{G_y}(z))du=\int_{-L}^{x}(S_F(u+iV)-S_{G_y}(u+iV))du$$
$$+\int_{v'}^{V}(S_F(-L+iu)-S_{G_y}(-L+iu))du$$
$$-\int_{v'}^{V}(S_F(x+iu)-S_{G_y}(x+iu))du. \tag{4.13}$$

Denote by $\xi(\eta)$ a random variable with distribution function $F(x)\,(G_y(x))$. Then we have

$$|S_F(-L+iv')| = \left|\mathbf{E}\frac{1}{\xi+L-iv'}\right| \le v'^{-1}\Pr\{|\xi|>L/2\} + \frac{2}{L}. \tag{4.14}$$

Similarly,

$$|S_{G_y}(-L+iv')| \le v'^{-1}\Pr\{|\eta|>L/2\} + \frac{2}{L}. \tag{4.15}$$

These inequalities imply that

$$\left|\int_{v'}^{V}(S_F(-L+iu)-S_{G_y}(-L+iu))du\right| \to 0 \quad \text{as} \quad L\to\infty, \tag{4.16}$$

which completes the proof. □

Combining the results of Proposition 4.1 and Corollary 4.1, we get

Corollary 4.2. *Under the conditions of Proposition 4.1 the following inequality holds*

$$\begin{aligned}\Delta(F,G) \le C_1\int_{-\infty}^{\infty}|S_F(u+iV)-S_{G_y}(u+iV)|du + C_2 v + C_3\varepsilon^{\frac{3}{2}}\\ + C_1\sup_{x\in\mathbb{J}'_\varepsilon}\int_{v'}^{V}|S_F(x+iu)-S_{G_y}(x+iu)|du,\end{aligned} \tag{4.17}$$

where $v' = \frac{v}{\sqrt{\gamma}}$ *with* $\gamma = \min\{|x|-1+\sqrt{y},\, 1+\sqrt{y}-|x|\}$.

We apply now the result of Corollary 4.2 to the empirical spectral distribution function $\mathcal{F}_n(x)$ of the random matrix $\mathbf{X}$. First we bound the integral over the line with $V=2$. Note that in this case we have $|z+ym_n(z)+\frac{y-1}{z}|\ge 1$. Moreover, $\operatorname{Im} m_n^{(j)}(z) \le \frac{1}{V} \le \frac{1}{2}$. We may now apply results of the previous Lemmas on large deviations. This ensures the following bound for $g_n(z)$ for all $z=u+iV$ with $u\in\mathbb{R}$.

$$|g_n(z)| \le \frac{\beta_n}{n\sqrt{v}|a_n(z,y)b_n(z,y)|}\Big(1+\frac{\beta_n\operatorname{Im} m_n(z)}{\sqrt{v}}+\frac{\beta_n}{nv}+\frac{(1-y)v}{|z|^2}\Big)+\frac{C\operatorname{Im} m_n(z)}{nv|a_n(z,y)|}. \tag{4.18}$$

Note that for $V=2$,

$$|a_n(z,y)b_n(z,y)| \ge \begin{cases} 4 & \text{for} \quad |u|\in[1-\sqrt{y},1+\sqrt{y}],\\ \frac{1}{4}|z|^2 & \text{for } |u|\notin[1-\sqrt{y},1+\sqrt{y}].\end{cases} \tag{4.19}$$

We may rewrite the bound (4.18) as follows

$$|g_n(z)| \le \frac{C\beta_n^2}{n(|z|^2+1)} + \frac{C\operatorname{Im} m_n(z)}{nv}. \tag{4.20}$$

Note that for any distribution function $F(x)$ we have

$$\int_{-\infty}^{\infty} \operatorname{Im} s_F(u+iv)du = \pi \tag{4.21}$$

From here it follows that, for $V = 2$

$$\int_{|u|\geq n} |m_n(z) - s_y(z)|du \leq \frac{C}{n} \tag{4.22}$$

Denote $\overline{\mathcal{D}}_n := \{z = u + 2i : |u| \leq n\}$ and

$$\overline{\Omega}_n := \Big(\cap_{z\in\overline{\mathcal{D}}_n} \{\omega \in \Omega : \ |g_n(z)| \leq \frac{C\beta_n^2}{n(|z|^2+1}\}\Big) \cap \Omega_n^*$$

Using a union bound, we may show that

$$\Pr\{\overline{\Omega}_n\} \geq 1 - C\exp\{-cl_{n,\alpha}\}. \tag{4.23}$$

It is straightforward to check that for $\omega \in \overline{\Omega}_n$

$$\int_{-\infty}^{\infty} |m_n(z) - s_y(z)|du \leq \frac{C}{n}. \tag{4.24}$$

We choose $\varepsilon = n^{-\frac{2}{3}}$ and $v_0 = \frac{d\beta_n^2}{n}$. To conclude the proof we need to consider the "vertical" path integrals in $z = x + iv'$ with $x \in \mathbb{J}'_\varepsilon$, $v' = \frac{v_0}{\sqrt{\gamma}}$ and $\gamma = 2 - |x|$. It will be enough to consider one of these integrals only, the others being similar, namely

$$\int_{v'}^{2} \frac{1}{n^2v^2\sqrt{\gamma+v}}dv \leq \frac{1}{n^2v'\sqrt{\gamma}} \leq \frac{1}{n^2v_0} \leq \frac{\beta_n^2}{n}. \tag{4.25}$$

Finally, we obtain for any $\omega \in \overline{\Omega}_n$

$$\Delta(F_n, G) = \sup_x |F_n(x) - G_y(x)| \leq \frac{C\beta_n^2}{n}. \tag{4.26}$$

Thus Theorem 1.1 is proved.

5 Proof of Theorem 1.2

Consider the singular value decomposition of the matrix $\mathbf{X}$. Let $\mathbf{U}$ and $\mathbf{H}$ be unitary matrices of dimension $n \times n$ and $p \times p$ respectively. Let $\mathbf{S}$ be a $n \times n$ diagonal matrix whose entries are the singular value of the matrix $\mathbf{X}$. Let $\mathbf{O}_{p\times q}$ denote the $p\times q$-matrix with zero entries. Introduce the matrix $\widetilde{\mathbf{S}} = \begin{bmatrix}\mathbf{S} & \mathbf{O}_{n\times(p-n)}\end{bmatrix}$. We have the following representation

$$\mathbf{X} = \mathbf{U}\widetilde{\mathbf{S}}\mathbf{H}^*. \tag{5.1}$$

We may represent the matrix $\mathbf{H}$ in the form

$$\mathbf{H} = \begin{bmatrix} \mathbf{H}_{11} & \mathbf{H}_{12} \\ \mathbf{H}_{21} & \mathbf{H}_{22} \end{bmatrix}, \tag{5.2}$$

where $\mathbf{H}_{11}$ is $n \times n$ matrix , $\mathbf{H}_{22}$ is $p \times p$ matrix. We introduce matrix

$$\mathbf{Z} = \begin{bmatrix} \frac{1}{\sqrt{2}}\mathbf{U} & \frac{1}{\sqrt{2}}\mathbf{H}_{11} & \frac{1}{\sqrt{2}}\mathbf{H}_{12} \\ \frac{1}{\sqrt{2}}\mathbf{U} & \frac{1}{\sqrt{2}} - \mathbf{H}_{11} & \frac{1}{\sqrt{2}} - \mathbf{H}_{12} \\ \mathbf{O} & \mathbf{H}_{21} & \mathbf{H}_{22} \end{bmatrix}. \tag{5.3}$$

It is straightforward to check that

$$\mathbf{Z}\mathbf{V}\mathbf{Z}^* = \begin{bmatrix} \mathbf{S} & \mathbf{O}_{n\times n} & \mathbf{O}_{n\times(p-n)} \\ \mathbf{O}_{n\times n} & -\mathbf{S} & \mathbf{O}_{n\times(p-n)} \\ \mathbf{O}_{(p-n)\times n} & \mathbf{O}_{(p-n)\times n} & \mathbf{O}_{(p-n)\times(p-n)} \end{bmatrix}, \tag{5.4}$$

where $\mathbf{S}$ denotes diagonal matrix with entries s_j. The equality (5.4) implies that the rows $\mathbf{z}_j$ of the matrix $\mathbf{Z}$, for $j = 1, \ldots, n$, are the eigenvectors of the matrix $\mathbf{V}$ corresponding to the eigenvalues s_j. Similarly, the rows $\mathbf{z}_{j+n}$ of the matrix $\mathbf{Z}$, for $j = 1, \ldots, n$, are the eigenvectors of the matrix $\mathbf{V}$ corresponding to the eigenvalues $-s_j$ and the rows $\mathbf{z}_{2n+l}$, for $l = 1, \ldots, p-n$, are the eigenvectors of matrix $\mathbf{V}$ corresponding to the eigenvalues 0.

We note the following representation for the diagonal entries of the resolvent matrix $\mathbf{R}$:

$$R_{jj} = \sum_{k=1}^{n+p} \frac{1}{\lambda_k - z} |H_{kj}|^2. \tag{5.5}$$

Denote by $\lambda_1, \ldots, \lambda_{n+p}$ the eigenvalues of the matrix $\mathbf{V}$ ordered in such way that

$$\lambda_j = \begin{cases} -s_j, & \text{if} \quad 1 \le j \le n \\ s_j, & \text{if} \quad n+1 \le j \le 2n \\ 0, & \text{if} \quad 2n \le j \le n+p. \end{cases} \tag{5.6}$$

Consider the distribution function $F_{nj}(x)$ of the following weighted empirical probability distribution on the eigenvalues $\lambda_1, \ldots, \lambda_{n+p}$

$$F_{nj}(x) = \sum_{k=1}^{n+p} |H_{kj}|^2 \mathbb{I}\{\lambda_k \le x\}. \tag{5.7}$$

Then we have

$$R_{jj} = R_{jj}(z) = \int_{-\infty}^{\infty} \frac{1}{x - z} dF_{nj}(x). \tag{5.8}$$

which means that R_{jj} is the Stieltjes transform of the distribution $F_{nj}(x)$. Note that, for any $\lambda > 0$

$$\max_{1 \le k \le n+p} |H_{kj}|^2 \le \sup_x (F_{nj}(x + \lambda) - F_{nj}(x)) =: Q_{nj}(\lambda). \tag{5.9}$$

On the other hand, it is easy to check that

$$Q_{nj}(\lambda) \le 2 \sup_u \lambda \mathrm{Im}\, R_{jj}(u+i\lambda). \tag{5.10}$$

By relations (2.30) and (2.34), for any $v \ge v_0$ with $v_0 = \frac{d\beta_n}{n}$ with a sufficiently large constant d, we have

$$\Pr\Big\{\frac{|\varepsilon_j|}{|b_n(z,y)|} > \frac{1}{2}\Big\} \le C\exp\{-cl_{n,\alpha}\}. \tag{5.11}$$

Furthermore, the representation (2.4) and inequality (5.11) together imply, for $v \ge v_0$

$$\mathrm{Im}\, R_{jj} \le |R_{jj}| \le C. \tag{5.12}$$

This implies that

$$\Pr\{\max_{1\le k\le n+p} |H_{kj}|^2 > \frac{C\beta_n^2}{n}\} \le C\exp\{-cl_{n,\alpha}\}. \tag{5.13}$$

By a union bound the inequality (1.4) follows. To prove inequality (1.5), we consider the quantity

$$r_j := R_{jj} - s_y(z), \quad j = 1,\ldots,n. \tag{5.14}$$

Using equalities (2.4) and (3.34), we get

$$r_j = -\frac{s_y(z)g_n(z)}{b_n(z,y)} + \frac{\varepsilon_j}{b_n(z,y)}R_{jj}. \tag{5.15}$$

By definition of $\mathbf{H}$ we obtain

$$\Pr\{\max_{1\le j,k\le n} |u_{kj}|^2 > \frac{C\beta_n}{n}\} \le C\exp\{-cl_{n,\alpha}\}. \tag{5.16}$$

and

$$\Pr\{\max_{1\le j,k\le p} |v_{kj}|^2 > \frac{C\beta_n}{n}\} \le C\exp\{-cl_{n,\alpha}\}. \tag{5.17}$$

By inequalities (3.62) and (2.34), we have

$$|r_j| \le \frac{c\beta_n}{\sqrt{nv}}. \tag{5.18}$$

This implies that

$$\sup_{x\in\mathbb{J}_\varepsilon} \int_{v'}^{V} |r_j(x+iv)|dv \le \frac{C}{\sqrt{n}}. \tag{5.19}$$

Similar to (4.24) we get

$$\int_{-\infty}^{\infty} |r_j(x+iV)|dx \le \frac{C}{\sqrt{n}}. \tag{5.20}$$

Applying Corollary 4.2, we finally obtain

$$\Pr\{\sup_x |F_{nj}(x) - G_y(x)| \le \frac{\beta_n}{\sqrt{n}}\} \ge 1 - C\exp\{-cl_{n,\alpha}\}. \tag{5.21}$$

In view of

$$\Pr\{\sup_x |F_n(x) - G_y(x)| \le \frac{\beta_n^2}{n}\} \ge 1 - C\exp\{-cl_{n,\alpha}\}, \tag{5.22}$$

we get

$$\Pr\{\sup_x |F_{nj}(x) - G_y(x)| \le \frac{\beta_n}{\sqrt{n}}\} \ge 1 - C\exp\{-cl_{n,\alpha}\}. \tag{5.23}$$

The last two inequalities together imply that

$$\Pr\{\sup_x |F_{nj}(x) - F_n(x)(x)| \le \frac{\beta_n}{\sqrt{n}}\} \ge 1 - C\exp\{-cl_{n,\alpha}\}. \tag{5.24}$$

Note that $F_n(x)$ is the distribution function of a random variable which is uniformly distributed on the set $\{\pm s_1, \ldots, \pm s_n\}$ and

$$\sup_x |F_{nj}(x) - F_n(x)| = \max_k \Big|\sum_{l=1}^{k} |u_{kj}|^2 - \frac{l}{n}\Big|. \tag{5.25}$$

Thus Theorem 1.2 is proved.

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